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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF $\Delta\mathbf{u}+\mathbf{g}(\mathbf{r})\mathbf{u}+\mathbf{h}(\mathbf{r})\mathbf{u}^{\mathbf{p}}=0$ AND ITS APPLICATIONS

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1. Introduction and main results

We consider the problem

$$\tag{1.1} \begin{cases} u_{rr}+\dfrac{n-1}{r}u_r+g(r)u+h(r)u^p=0, & 0<r<R, \\ u(0)\in(0,\infty), \quad u(R)=0, \end{cases}$$

where $n\geq 2$, $R\in(0,\infty]$, $p\in(1,\infty)$ and $g,h:(0,R)\to\mathbb{R}$ are appropriate functions. Here, $u(R)=0$ in the case $R=\infty$ means $u(x)\to 0$ as $|x|\to\infty$. Such a problem has been studied by many researchers; see [1,3,5,8,9,12–18,20–27,30,32–36] and others.

In this note, we introduce a result obtained in [28].

Theorem 1. *Let $0<R\leq\infty$, $n\in\mathbb{R}$ with $n\geq 2$ and $p\in(1,\infty)$. Let $g\in C([0,R))\cap C^1((0,R))$ and $h\in C^2([0,R))\cap C^3((0,R))$ such that h is positive on $[0,R)$. with $R'=0$. Assume the following.*

(i) *In the case of $R<\infty$, $g\in C([0,R])$, $h\in C^2([0,R])$ and $h(R)>0$ are also satisfied.*
(ii) *There exists $\kappa\in[0,R]$ such that*

$$G(r)\geq 0 \text{ on } (0,\kappa) \quad\text{and}\quad G(r)\leq 0 \text{ on } (\kappa,R),$$

where

$$\begin{aligned} G(r)=\frac{r^{\frac{2(n-1)(p+1)}{p+3}-3}}{2(p+3)^3h(r)^{\frac{2}{p+3}+3}}\Big(&4(n-1)\big[n+2-(n-2)p\big]\big[n-4+(n-2)p\big]h(r)^3 \\ &+\Big[2(n-1)(p-1)(p+3)^2r^2h(r)^3-4(p+3)^2r^3h(r)^2h_r(r)\Big]g(r) \\ &+(p+3)^3r^3g_r(r)h(r)^3 \\ &+(n-1)\big[(2n-3)p(6-p)+6n-33\big]rh(r)^2h_r(r) \\ &+3(n-1)(p-1)(p+5)r^2h(r)h_r(r)^2-2(p+4)(p+5)r^3h_r(r)^3 \end{aligned}$$

$$- 3(n-1)(p-1)(p+3)r^2h(r)^2h_{rr}(r)$$

$$+ 3(p+3)(p+5)r^3h(r)h_r(r)h_{rr}(r) - (p+3)^2r^3h(r)^2h_{rrr}(r)\Big).$$

(iii) *In the case of* $R=\infty$, $G^-\not\equiv 0$ *is satisfied.*

Then in the case of $R<\infty$, *problem* (1.1) *has at most one positive solution, and in the case of* $R=\infty$, *problem* (1.1) *has at most one positive solution* u *which satisfies* $J(r;u)\to 0$ *as* $r\to\infty$, *where*

$$a(r) = r^{\frac{2(n-1)(p+1)}{p+3}}h(r)^{\frac{-2}{p+3}},$$

$$b(r) = \frac{r^{\frac{2(n-1)(p+1)}{p+3}-1}}{(p+3)h(r)^{\frac{p+5}{p+3}}}\big(2(n-1)h(r)+rh_r(r)\big),$$

$$c(r) = \frac{r^{\frac{2(n-1)(p+1)}{p+3}-2}}{(p+3)^2h(r)^{\frac{2(p+4)}{p+3}}}\Big(2(n-1)\big[n+2-(n-2)p\big]h(r)^2 + (p+5)r^2h_r(r)^2$$

$$- (n-1)(p-5)rh(r)h_r(r) - (p+3)r^2h(r)h_{rr}(r)\Big),$$

$$J(r;u) = \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(r)u(r) + \frac{1}{2}c(r)u(r)^2$$

$$+ \frac{1}{2}a(r)g(r)u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1}.$$

Remark 1. In [32, Theorems 2.1 and 2.2], Yanagida obtained a closely related result.

By the theorem above, we can obtain the following; see [13, Theorem 0.1].

Corollary 1 (Kabeya-Tanaka). *Let* $n\in\mathbb{N}$ *with* $n\geq 2$. *Let* $p>1$ *and* $g\in C^2([0,\infty))$ *such that* $-\infty<\inf_{r\in[0,\infty)}g(r)\leq\sup_{r\in[0,\infty)}g(r)<0$, *and set*

$$L = \frac{2(n-1)[(n-2)p+n-4]}{(p+3)^2} \quad \text{and} \quad \beta = \frac{2(n-1)(p-1)}{p+3}.$$

Assume that

$$g_r(r)r^3 + \beta g(r)r^2 - (\beta-2)L < 0 \quad \text{for each } r\geq 0$$

in the case of $n=2$, *and that* $p<(n+2)/(n-2)$ *and*

$$\sup_{r>0}\big(g_{rr}(r)r^2 + (3+\beta)g_r(r)r + 2\beta g(r)\big) < 0$$

in the case of $n\geq 3$. *Then the problem*

$$u\in H^1(\mathbb{R}^n), \qquad \Delta u(x) + g(|x|)u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n \tag{1.2}$$

has a unique positive radial solution.

Next, we consider the problem

$$\text{(1.3)} \qquad \begin{cases} u_{rr}(r) + \dfrac{n-1}{r}u_r + g(r)u(r) + h(r)u(r)^p = 0, & R' < r < R, \\ u(R') = 0, \quad u(R) = 0. \end{cases}$$

The uniqueness of a positive solution of such a problem was studied in [4, 6, 7, 10, 11, 19, 24, 29–31].

The following is also obtained in [28].

Theorem 2. *Let $0 < R' < R \leq \infty$, $n \in \mathbb{R}$, $p \in (1, \infty)$, $g \in C([R', R)) \cap C^1((R', R))$, $h \in C^2([R', R)) \cap C^3((R', R))$ such that h is positive on $[R', R)$. Let a, b, c, G and J be the functions given in Theorem 1. Assume the following.*

(i) *In the case of $R < \infty$, $g \in C([R', R])$, $h \in C^2([R', R])$ and $h(R) > 0$ are also satisfied.*

(ii) *There exists $\kappa \in [R', R]$ such that*

$$G(r) \geq 0 \text{ on } (R', \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R).$$

Then in the case of $R < \infty$, problem (1.3) has at most one positive solution, and in the case of $R = \infty$, problem (1.3) has at most one positive solution u which satisfies $J(r; u) \to 0$ as $r \to \infty$.

Remark 2. For the case $h(r) \equiv 1$, a similar result is obtained by Felmer-Martínez-Tanaka; see [10, Theorem 1.1].

2. Applications

In this section, we give examples of Theorem 1. First, we give a comment on the scalar field equation

$$\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty.$$

The unique existence of its positive solution was established by Kwong [18]. Since the uniqueness of its positive solution can be derived from Corollary 1, of course, it can be also done by Theorem 1.

Next, we consider the following Brezis-Nirenberg problem.

$$\text{(2.1)} \qquad \begin{cases} \Delta_{S^n} u + \lambda u + u^p = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Here, n is a natural number with $n \geq 3$, S^n is the unit sphere in $\mathbb{R}^{n+1}$, Δ_{S^n} is the Laplace-Beltrami operator on S^n, $D = \{X \in S^n : X_{n+1} > \cos\theta_1\}$ with $\theta_1 \in (0, \pi)$,

$1 < p \leq (n+2)/(n-2)$ and $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta_{S^n}$ on D with the Dirichlet boundary condition.

Let $P : S^n \setminus \{(0,\dots,0,-1)\} \to \mathbb{R}^n$ be the stereographic projection defined by

$$P(X_1,\dots,X_n,X_{n+1}) = \frac{1}{X_{n+1}+1}(X_1,\dots,X_n) \quad \text{for } X \in S^n \setminus \{(0,\dots,0,-1)\}.$$

Then we can see $P(D) = B_R$, where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ with

$$R = \frac{\sin\theta_1}{1+\cos\theta_1}.$$

Let u be a positive solution of (2.1) and define $v : \overline{B_R} \to \mathbb{R}$ by $u(P^{-1}x) = (1+|x|^2)^{\frac{n-2}{2}} v(x)$ for $x \in \overline{B_R}$. Then we see that v is a positive solution of

$$\begin{cases} \Delta v + \dfrac{n(n-2)+4\lambda}{(1+|x|^2)^2} v + 4(1+|x|^2)^{\frac{(n-2)p-(n+2)}{2}} v^p = 0 & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R. \end{cases}$$

We set

$$g(r) = \frac{n(n-2)+4\lambda}{(1+r^2)^2} \quad \text{and} \quad h(r) = 4(1+r^2)^{\frac{(n-2)p-(n+2)}{2}} \quad \text{for } r \geq 0.$$

We can see that G in Theorem 1 is given by

$$G(r) = \frac{2^{\frac{p-1}{p+3}}(n-1)}{(p+3)^3} r^{\frac{2(n-1)(p+1)}{p+3}-3}(1+r^2)^{\frac{n+2-(n-2)p}{p+3}-3}(1-r^2)(Ar^4+Br^2+A),$$

where

$$\begin{aligned} A &= (n-2)^2\left(\frac{n+2}{n-2} - p\right)\left(p + \frac{n-4}{n-2}\right) \\ &= (p+3)[3n^2 - 6n - (n^2-4n+4)p] - 8(n-1)^2, \\ B &= (p+3)[-6n^2 + 12n + (2n^2+4\lambda-4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n-1)^2. \end{aligned}$$

Then we can infer the following. For the details, see [28].

Theorem 3. *Let $n \in \mathbb{N}$ with $n \geq 3$, $1 < p \leq (n+2)/(n-2)$ and $\theta_1 \in (0,\pi)$. Assume that one of the following conditions:*

(i) $\theta_1 \in (0,\pi/2]$ *and* $\lambda < \lambda_1$,

(ii) $\theta_1 \in (\pi/2,\pi)$ *and*

$$\frac{6+(6-4n)p}{(p+3)(p-1)} \leq \lambda < \lambda_1.$$

Then (2.1) has at most one positive radial solution. Moreover, if $\lambda \geq -n(n-2)/4$ is also satisfied, then (2.1) has at most one positive solution.

Remark 3. It holds that
$$\frac{6+(6-4n)p}{(p+3)(p-1)} \leq -\frac{n(n-2)}{4},$$
and if $p = (n+2)/(n-2)$ then the constants in the both sides in the inequality above coincide.

Remark 4. In the case of $n = 3$, Bandle-Benguria obtained a sharper result. For the details, see [2].

Remark 5. In the case of $R > 1$, we cannot apply Yanagida's uniqueness theorem [32, Theorem 2.1]. Indeed, by his notation, we have
$$G(r; n-2) = \frac{2(4\lambda + n(n-2))r^{n-1}(1-r^2)}{(r^2+1)^3}.$$
So one of his assumptions $G(r; n-2) \leq 0$ on $(0, R)$ is not satisfied even if $\lambda > -n(n-2)/4$.

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